Solutions to Practice Problems

Exercise 3.7
Consider the set $A = \{\frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Solution.
(a) Clearly, $\frac{1}{2}$ is an upper bound of A. Let $M > 0$ be an upper bound of A. We will show that $\frac{1}{2} \le M$. Suppose the contrary. That is, suppose that $M < \frac{1}{2}$. Since M is an upper bound of A, we have $\frac{(-1)^n}{n} \le M$ for all $n \in \mathbb{N}$. In particular, letting $n = 2$ we obtain $\frac{1}{2} \le M < \frac{1}{2}$ which is imposssibe. Thus, $\frac{1}{2} \le M$ so that $\sup\{A\} = \frac{1}{2}$. Since the supremum is an element of A we conclude that $\frac{1}{2}$ is also the maximum of A.
(b) Clearly, -1 is a lower bound of A. Let m be a lower bound of A. We will show that $m \le -1$. Suppose the contrary. That is, suppose that $m > -1$. Letting $n = 1$ we find that $-1 = \frac{(-1)}{1} \ge m > -1$, which is impossible. Therefore, we must have $m \le -1$. This establishes that $\inf\{A\} = -1$. Since -1 is in A, it is the minimum of A ■

Exercise 3.8
Consider the set $A = \{x \in \mathbb{R} : 1 < x < 2\}$.
(a) Show that A is bounded from above. Find the supremum. Is this supremum a maximum of A?
(b) Show that A is bounded from below. Find the infimum. Is this infimum a minimum of A?

Solution.
(a) Clearly, 2 is an upper bound of A. Let $M > 1$ be an upper bound of A. We will show that $2 \le M$. Suppose the contrary. That is, suppose that $1 < M < 2$. Let r be a rational number such that $M < r < 2$. Then $r \in A$ and $M < r$ which contradicts the fact that M is an upper bound of A. Hence, we must have $2 \le M$ so that $\sup\{A\} = 2$. Since the supremum is not an element of A we conclude that 2 is not a maximum of A.
(b) Clearly, 1 is a lower bound of A. Let m be a lower bound of A. We will show that $m \le 1$. Suppose the contrary. That is, suppose that $1 < m < 2$.

Let r be a rational number such that $1 < r < m$. Then $r \in A$ and $r < m$ which contradicts the fact that m is a lower bound of A. Thus, we must have $1 \leq m$ so that $\inf\{A\} = 1$. Since 1 is not in A, it is not a minimum of A ■

Exercise 3.9
Consider the set $A = \{x > 0 : x^2 > 4\} = \{x > 0 : x > 2\}$.
(a) What is a lower bound of A?
(b) Let L be a lower bound of A such that $L > 2$. Let $y = \frac{L+2}{2}$. Show that $2 < y < L$.
(c) Show that $y \in A$ and $L \leq y$. Show that this leads to a contradiction. Hence, we must have $L \leq 2$ which means that 2 is the infimum of A.

Solution.
(a) Since $2 \leq x$ for all $x \in A$, 2 is a lower bound of A.
(b) Since $L > 2$ we have $L + 2 > 4$ and this implies $y = \frac{L+2}{2} > 2$. Also, $y = \frac{L+2}{2} < \frac{L+L}{2} = L$.
(c) Since $y > 2$ we have $y^2 > 4$ so that $y \in A$. But L is a lower bound of A so we must have $L \leq y$. But this contradicts $y < L$ from (b). It follows that 2 is the greatest lower bound of A ■

Exercise 3.10
Show that for any real number x there is a positive integer n such that $n > x$.

Solution.
Let $a = 1$ and $b = x$ in the Archimedean property ■

Exercise 3.11
Let a and b be any two real numbers such that $a < b$.
(a) Let w be a fixed positive irrational number. Show that there is a rational number r such that $a < wr < b$.
(b) Show that wr is irrational. Hence, between any two distinct real numbers there is an irrational number.

Solution.
(a) Since $a < b$, we have $\frac{a}{w} < \frac{b}{w}$. By Exercise 3.6, there is a rational number r such that $\frac{a}{w} < r < \frac{b}{w}$ or $a < rw < b$.
(e) If $rw = s$ with s rational then $w = \frac{s}{r}$ which is a rational, a contradiction. Hence, rw is irrational ■

Exercise 3.12
Suppose that $\alpha = \sup A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\alpha - \epsilon < x$.

Solution.
Suppose the contrary. That is, $\alpha - \epsilon \geq x$ for all $x \in A$. In this case, $\alpha - \epsilon$ is an upper bound of A. Thus, we must have $\alpha \leq \alpha - \epsilon$ which is impossible ■

Exercise 3.13
Suppose that $\beta = \inf A < \infty$. Let $\epsilon > 0$ be given. Prove that there is an $x \in A$ such that $\beta + \epsilon > x$.

Solution.
Suppose the contrary. That is, $\beta + \epsilon \leq x$ for all $x \in A$. In this case, $\beta + \epsilon$ is a lower bound of A. Thus, we must have $\beta + \epsilon \leq \beta$ which is impossible ■

Exercise 3.14
For each of the following sets S find $\sup\{S\}$ and $\inf\{S\}$ if they exist. You do not need to justify your answer.
(a) $S = \{x \in \mathbb{R} : x^2 < 5\}$.
(b) $S = \{x \in \mathbb{R} : x^2 > 7\}$.
(c) $S = \{-\frac{1}{n} : n \in \mathbb{N}\}$.

Solution.
(a) Note that $S = \{x \in \mathbb{R} : -\sqrt{5} < x < \sqrt{5}\}$. So $\sqrt{5}$ is an upper bound of S. Let M be an upper bound of S. Suppose that $M < \sqrt{5}$. Let r be a rational number such that $M < r < \sqrt{5}$. Then $r \in S$ and $M < r$. But this contradicts the fact that M is an upper bound of S. Thus, $\sqrt{5} \leq M$ so that $\sup\{S\} = \sqrt{5}$.
Likewise one can show that $\inf\{S\} = -\sqrt{5}$.
(b) $\sup\{S\} = \infty$ and $\inf\{S\} = -\infty$.
(c) $\sup\{S\} = 0$ and $\inf\{S\} = -1$ ■

Exercise 3.15
(a) Show that for any positive numbers a and b we have $\frac{a+b}{2} \geq \sqrt{ab}$.
(b) Let $a_i > 0$ for $i = 1, 2, \cdots, n$. Suppose that $\sqrt[n]{a_1 a_2 \cdots a_n} = 1$. Use (a) to show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq 2^n.$$

Solution.
(a) We have $(\sqrt{a}-\sqrt{b})^2 \geq 0 \to a+b \geq 2\sqrt{ab} \to \frac{a+b}{2} \geq \sqrt{ab}$.
(b) For $i = 1, \cdots, n$ we have $\frac{1+a_i}{2} \geq \sqrt{a_i}$. Multiplying these n inequalitites we find

$$\left(\frac{1+a_1}{2}\right)\left(\frac{1+a_2}{2}\right)\cdots\left(\frac{1+a_n}{2}\right) \geq \sqrt{a_1a_2\cdots a_n} = 1$$

Hence,

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 2^n \blacksquare$$

Exercise 3.16
Consider the numbers $s_1, s_2, \cdots$ where $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2+s_n}$ for $n \in \mathbb{N}$. Show that each of these numbers is irrational.

Solution.
We proof this claim by induction on n. For $n = 1$ we have $s_1 = \sqrt{2}$ which is an irrational number. Suppose that s_k is irrational for $k = 1, 2, \cdots, n$. We want to show that s_{n+1} is irrational. Suppose the contrary, then $s_n = s_{n+1}^2 - 2$ is rational which contradicts the assumption that s_n is irrational. Hence, s_{n+1} must be irrational $\blacksquare$

Exercise 3.17
An **algebraic number** is a number that satisfies a polynomial equation with integer coefficients. Show that $x = \sqrt[3]{1+\sqrt{5}}$ is an algebraic number.

Solution.
We have $x = \sqrt[3]{1+\sqrt{5}} \Longrightarrow x^3 = 1+\sqrt{5} \Longrightarrow (x^3-1)^2 = 5 \Longrightarrow x^6 - 2x^3 - 4 = 0$ $\blacksquare$

Exercise 3.18
Let $A \subset \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ be such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in A$. Show the following
(a) $\sup\{f(x)+g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}$.
(b) $\inf\{f(x)+g(x) : x \in A\} \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}$.
(c) $\sup\{-f(x) : x \in A\} = -\inf\{f(x) : x \in A\}$
(d) $\sup\{f(x)-g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} - \inf\{g(x) : x \in A\}$.

Solution.
(a) For all $x \in A$, we have

$$f(x) + g(x) \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}.$$

Thus, $\sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}$ is an upper bound of $\{f(x) + g(x) : x \in A\}$. But $\sup\{f(x) + g(x) : x \in A\}$ is the smallest upper bound of $\{f(x) + g(x) : x \in A\}$ so that

$$\sup\{f(x) + g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}.$$

(b) Similar argument to (a).
(c) We have $-f(x) \leq -\inf\{f(x) : x \in A\}$ for all $x \in A$ so that $-\inf\{f(x) : x \in A\}$ is an upper bound of $\{-f(x) : x \in A\}$. But $\sup\{-f(x) : x \in A\}$ is the smallest upper bound of $\{-f(x) : x \in A\}$. Hence,

$$\sup\{-f(x) : x \in A\} \leq -\inf\{f(x) : x \in A\}.$$

Suppose that $\sup\{-f(x) : x \in A\} < -\inf\{f(x) : x \in A\}$. Let $\epsilon = -\inf\{f(x) : x \in A\} - \sup\{-f(x) : x \in A\} > 0$. By Exercise 3.13, there is a $a \in A$ such that $\inf\{f(x) : x \in A\} + \epsilon > f(a)$. But this implies that $-\inf\{f(x) : x \in A\} - \epsilon < -f(a) \leq \sup\{-f(x) : x \in A\}$. This leads to the contradiction $\sup\{-f(x) : x \in A\} < \sup\{-f(x) : x \in A\}$. Hence, the equality must hold.
(d) We have

$$\begin{aligned}
\sup\{f(x) - g(x) : x \in A\} =& \sup\{f(x) + (-g(x)) : x \in A\} \\
\leq& \sup\{f(x) : x \in A\} + \sup\{-g(x) : x \in A\} \\
=& \sup\{f(x) : x \in A\} - \inf\{g(x) : x \in A\} \blacksquare
\end{aligned}$$

Exercise 3.19
Let $A \subset \mathbb{R}$. Let $f, g : A \to \mathbb{R}$ be such that $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$ for all $x \in A$.
(a) Show that

$$\{f(x) - g(y) : x, y \in A\} = \{|f(x) - g(y)| : x, y \in A\} \cup \{-|f(x) - g(y)| : x, y \in A\}.$$

(b) Show that

$$\sup\{|f(x) - g(y)| : x, y \in A\} = \sup\{f(x) - g(y) : x, y \in A\}.$$

Solution.
(a) Let $x, y \in A$. If $f(x) \geq f(y)$ then $f(x) - f(y) = |f(x) - f(y)| \in \{|f(x) - g(y)| : x, y \in A\}$. If $f(x) \leq f(y)$ then $f(x) - f(y) \leq 0$ so that $f(x) - f(y) = -|f(x) - f(y)| \in \{-|f(x) - g(y)| : x, y \in A\}$. Hence,

$$\{f(x)-g(y) : x, y \in A\} \subseteq \{|f(x)-g(y)| : x, y \in A\} \cup \{-|f(x)-g(y)| : x, y \in A\}.$$

The other inclusion is trivial.
(b) Note that $\{|f(x) - g(y)| : x, y \in A\}$ consists of nonnegative numbers whereas $\{-|f(x) - g(y)| : x, y \in A\}$ consists of negative numbers. Hence, the result is clear ■

Exercise 3.20
(a) For each $n \in \mathbb{N}$ we define $I_n = [n, \infty)$. Show that $\bigcap_{n=1}^{\infty} I_n = \emptyset$. Hint: Archimedean property.
(b) For each $n \in \mathbb{N}$ we define $J_n = [-\frac{1}{n}, \infty)$. Show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Solution.
(a) Let $a > 0$. By the Archimedean property, there is a positive integer n such that $n > y$. That is, $y \notin [n, \infty)$ and so $y \notin \bigcap_{n=1}^{\infty} I_n$. Since y was arbitrary, we must have $\bigcap_{n=1}^{\infty} I_n = \emptyset$.
(b) $0 \in \bigcap_{n=1}^{\infty} J_n$ ■

Exercise 3.21 *(Nested interval theorem)*
For each $n \in \mathbb{N}$ let $I_n = [a_n, b_n]$. Suppose that $I_{n+1} \subset I_n$ for each $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Solution.
(a) We have $a_1 \leq a_2 \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$. Hence, for each $n \in \mathbb{N}$, b_n is an upper bound of the set $\{a_1, a_2, \cdots\}$. By completeness of $\mathbb{R}$ there is a finite number α such that $\alpha = \sup\{a_1, a_2, \cdots\}$. Fix $n \in \mathbb{N}$. By the definition of α we have $a_n \leq \alpha$. Now, since b_n is an upper bound of $\{a_1, a_2, \cdots\}$, again by the definition of α we have $\alpha \leq b_n$. Hence, $\alpha \in I_n$ for all $n \in \mathbb{N}$ so that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ ■

Exercise 3.22
Show that $\sqrt{2} + \frac{1}{\sqrt{2}}$ is irrational.

Solution.
Suppose the contrary. Then $\sqrt{2} + \frac{1}{\sqrt{2}} = \frac{p}{q}$ where p and q are non-zero integers. Multiply through by $\sqrt{2}$ to obtain $3 = \sqrt{2}\frac{p}{q}$. Hence, $\sqrt{2} = \frac{3q}{p}$ which is a rational. But this contradicts the fact that $\sqrt{2}$ is irrational ■

Exercise 3.23 *(Cauchy-Schwarz inequality)*
Let $x_1, x_2, y_1, y_2 \in \mathbb{R}$.
(a) Show that $x^2 + y^2 \geq 2xy$ for all $x, y \in \mathbb{R}$.
(b) Show that

$$2\frac{x_1y_1}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_1^2}{x_1^2 + x_2^2} + \frac{y_1^2}{y_1^2 + y_2^2}$$

and

$$2\frac{x_2y_2}{\sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}} \leq \frac{x_2^2}{x_1^2 + x_2^2} + \frac{y_2^2}{y_1^2 + y_2^2}$$

(c) Show that

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

Solution.
(a) This follows from the fact that $(x - y)^2 \geq 0$.
(b) For the first inequality let $x = \frac{x_1}{\sqrt{x_1^2+x_2^2}}$ and $y = \frac{y_1}{\sqrt{y_1^2+y_2^2}}$. Likewise, for the second inequality, let $x = \frac{x_2}{\sqrt{x_1^2+x_2^2}}$ and $y = \frac{y_2}{\sqrt{y_1^2+y_2^2}}$.
(b) This follows by adding the two inequalities in (b) ■

Exercise 3.24
For each $n \in \mathbb{N}$ let $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Show that $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

Solution.
Clearly, $0 \in \bigcap_{n=1}^{\infty} I_n$. Suppose that $x \in \bigcap_{n=1}^{\infty} I_n$ with $x \neq 0$. If $x > 0$ then by the Archimedean property we can find a positive integer n such that $nx > 1$. But this implies that $x \notin \left(-\frac{1}{n}, \frac{1}{n}\right)$ and therefore $x \notin \bigcap_{n=1}^{\infty} I_n$. If $x < 0$ then $-x > 0$ and we can find a positive integer n such that $n(-x) > 1$ so that $x < -\frac{1}{n}$. Again this implies that $x \notin \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n$ consists of $\{0\}$ ■